Chapter 3
Theory of Single Filamentation

3.1 Introduction

The propagation of a femtosecond optical pulse is described by Maxwell’s equations. A number of linear and nonlinear effects such as self-focusing, dispersion, self-phase modulation, and ionization have to be taken into account. The overall dynamics of such pulses can be complicated where both transverse and temporal effects play equally important roles (Chin et al., 2005; Couairon and Mysyrowicz, 2007; Bergé et al., 2007 and references therein). Unfortunately, so far there is no analytical solution to the problem. Finding direct numerical solutions with a computer requires enormous computational efforts and in many cases does not provide an insight to the basic physical understanding of the various linear and nonlinear effects involved. Therefore, an approximate wave equation is used instead with a more reasonable mathematical solution. The interaction of intense optical pulses in a bulk medium is highly nonlinear and the material response must therefore couple self-consistently with the wave equation. The following is adapted from Chin et al. (2005).

3.2 Filamentation in Air

We consider propagation in air as a concrete example. From Maxwell’s equations it is possible to obtain a second order scalar wave equation for the electric field. This scalar equation is obtained by assuming that we have a linearly polarized electric field, \( E \), propagating in the medium which is assumed isotropic. The isotropic nature of the medium is assumed unchanged during filamentation. The vector nature of the field is thus suppressed in writing the expression with the understanding that it is linearly polarized. The derivation of this equation can be found in any book on electromagnetic theory. It reads (in Gaussian units)

\[
\frac{\partial^2 E}{\partial z^2} + \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} - \frac{4\pi}{c^2} \frac{\partial^2 P}{\partial t^2} - \frac{4\pi}{c^2} \frac{\partial J}{\partial t} = 0
\]  

(3.1)
where $P$ is the polarization response of the medium and includes both linear and nonlinear responses of the medium. The current density $J$ comes from free electrons created by field/tunnel ionization. It can be written as

$$J = -eN_e v_e. \quad (3.2)$$

where the electron velocity $v_e$ is derived from

$$\frac{\partial v_e}{\partial t} = -\frac{eE}{m_e}. \quad (3.3)$$

Let us take the time derivative of Eq. (3.2) and assuming that the initial electron velocity is zero when it is created, we obtain

$$\frac{\partial J}{\partial t} = \frac{e^2 N_e}{m_e} E. \quad (3.4)$$

Inserting Eq. (3.4) into Eq. (3.1) we obtain

$$\frac{\partial^2 E}{\partial z^2} + \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} - \frac{4\pi}{c^2} \frac{\partial^2 P}{\partial t^2} - \frac{4\pi e^2 N_e}{c^2 m_e} E = 0 \quad (3.5)$$

First we consider laser pulse propagation in the framework of the slowly varying envelope approximation (SVEA) and will discuss the effects of higher-order correction terms such as self-steepening later. We assume that the medium polarization is given by $P = \chi^{(1)} E + \chi^{(3)} E^3$, where $\chi^{(1)}$ and $\chi^{(3)}$ describe the linear and nonlinear susceptibility coefficients, respectively. The dielectric function is given as $\varepsilon = 1 + 4\pi \chi^{(1)}$. The electric field is assumed to have a rapidly oscillating part $e^{-i\omega t + ikz}$ that is modulated by an envelope given as $E(x,y,z,t) = \varepsilon(x,y,z,t)e^{-i\omega t + ikz} + c.c.$, where $\varepsilon(x,y,z,t)$ is assumed to be a slowly varying envelope function such that it varies slowly in time and space on the scales of $\omega^{-1}$ and $k^{-1}$. Inserting $P$ and $E$ into Eq. (3.5) and applying the slowly varying envelope approximation one obtains

$$i \frac{\partial \varepsilon}{\partial z} + \frac{1}{2k} \nabla^2_{\perp} \varepsilon + n_2 k_0 \varepsilon |\varepsilon|^2 \varepsilon - \frac{2\pi e^2 N_e}{km_e c^2} \varepsilon + i\Gamma \varepsilon = 0 \quad (3.6)$$

Here, $\nabla^2_{\perp} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian operator. In addition, for simplicity, effects arising from the group velocity dispersion are neglected and can be added easily as is done later. Equation (3.6) is the so-called nonlinear Schrödinger equation. The intensity of the pulse is defined as $I = |\varepsilon|^2$ given in units of W/cm$^2$, and $n_2 = 12\pi^2 \chi^{(3)}/n_0^2 c$ is the nonlinear coefficient given in units of cm$^2$/W and $\Gamma$ accounts for ionization losses. The generation and evolution of electron density $N_e(x,y,t)$ for single ionized molecules is given by

$$\frac{\partial N_e}{\partial \tau} = (N_0 - N_e)R(|\varepsilon|^2) \quad (3.7)$$
where $R$ is the multiphoton/tunnel ionization rate (in units of $s^{-1}$) for air molecules (oxygen and nitrogen) and $N_0$ is the number density of neutral molecules in units of molecules/cm$^3$. Effects such as electron recombination and cascade ionization, which do not play an important role for pulses shorter than 1 ps (Section 1.4), are neglected.

Equation (3.6) when coupled to Eq. (3.7) includes the most basic linear and nonlinear terms for describing self-focusing, diffraction, plasma generation and defocusing, leading to the concept of laser pulse filamentation. It is referred to as the nonlinear Schrödinger equation and has been studied widely in nonlinear optics. As we will show later, the slowly varying envelope approximation eventually breaks down but nevertheless provides an important physical understanding of these phenomena.

3.3 Numerical Solution of Filamentation in Air

We shall analyze the numerical solution and compare with known experimental results. We consider the propagation in air of an initially collimated Gaussian beam of the form $\varepsilon(r, \tau) = A_0 e^{-(r^2/w_0^2+\tau^2/\tau_0^2)}$, where $w_0$ and $\tau_0$ are the initial beam radius and pulse width respectively, measured at $1/e^2$ of the intensity (Fig. 3.1a). Equations (3.6) and (3.7) are integrated with $w_0 = 0.025$ cm, $\tau_0 = 85$ fs and $P_0 = 6P_{cr}$, where $P_{cr} = 3$ GW is the critical power for self-focusing in air at 800 nm. In addition, cylindrical symmetry of the transverse profile is assumed which is valid when describing the evolution of a single filament. In Fig. 3.1, the time scale is the local time of the pulse. Negative time means the front part of the pulse. We can also imagine that the pulse is propagating in space in the $z$-direction ($z = ct$ where $c$ is the speed of light and $t$ is time). In this case, $z$ is toward the direction of negative time. (It should be noted that the critical power for self-focusing is pulse-duration dependent in air because of the molecular delayed response (Liu and Chin, 2005). For about 100 fs or longer pulses, $P_{cr}$ is around 4 GW. It is higher for shorter pulses. It could be as high as 10 GW in the case of 45 fs pulses. In this chapter, the “old” value, 3 GW, is kept. It would not change the physical understanding of the physics).

Electron generation through multiphoton/tunnel ionization of nitrogen and oxygen molecules (assuming 80% nitrogen and 20% oxygen) is taken into account via the rates $R$ in Eq. (3.7). The rate $R$ is fitted by the empirical formula $R = \sigma^{(n)} I^n$, where $\sigma^{(n)}$ and $n$ are fitting parameters obtained through experimental measurement of the ionization of oxygen and nitrogen molecules separately using 200 fs Ti:sapphire laser pulses (Talebpour et al., 1999a). The experimental results are plotted on log-log scales of the number of ions generated through tunnel ionization vs. laser intensity. For the relevant intensity range up to about $10^{14}$ W/cm$^2$, the plot is a straight line whose slope indicates the exponent $n$ (Chin, 2004). Unlike the result of multiphoton perturbation theory, this $n$ is, in general, not an integer number because the ionization process is in the tunneling regime (Chin, 2004; Chin et al., 1985).
A more theoretical approach is to use the “Intense-Field Many-Body S-Matrix Theory” (IMST) to calculate the ionization rate (Becker and Faisal, 2005). This theory was found to agree very well with the experimental data obtained by Talebpour et al. (1999a). But since during filamentation, the pulse gets shorter and shorter, the ionization physics would evolve into the few-cycle regime where the experimental results might not be the same. However, the ionization rates obtained using the IMST are still valid as long as the pulse width at half the maximum intensity (FWHM) is at least three field cycles long (Becker et al., 2001b). (This is the reason why in the simulation described above, the experimental data are used for the sake of simplicity).

**Fig. 3.1** Evolution of a femtosecond laser pulse propagating in air. Shown is the spatio-temporal intensity distribution of (a) an initially Gaussian pulse propagating in ionizing air at \( z = 0 \), (b) \( z/z_0 = 0.4 \), (c) \( z/z_0 = 0.6 \), (d) \( z/z_0 = 0.8 \), and (e) \( z/z_0 = 1 \). The intensity is normalized to the peak input intensity of (a) and the radius and time coordinates are scaled to the initial beam radius \( w_0 = 0.025 \) cm, pulse width \( \tau_0 = 85 \) fs and \( P_0 = 6P_{cr} \), where \( P_{cr} = 3 \) GW (courtesy of Dr. Neset Aközbebek).
3.3 Numerical Solution of Filamentation in Air

Figure 3.1a–e gives the results of simulation in which the spatio-temporal intensity distribution is plotted at various propagation distances $z$ normalized to the diffraction length $z_0 = \frac{k w_0^2}{2}$ of the collimated input beam (Aközbek et al., 2001). As the pulse self-focuses, the peak intensity increases very rapidly until there is enough plasma to stop the focusing process. The peak (strongest part) of the pulse will come to a focus first and will be stopped by the defocusing of the self-induced plasma as shown in Fig. 3.1b (intensity clamping). Note again that the front part of the pulse is in the negative local time region. This is followed and repeated by other parts at the front part of the pulse; i.e., “slice-by-slice self-focusing” as discussed in Chapter 2. The plasma generation is a cumulative process and each slice experiences a different magnitude of plasma defocusing. Thus some of the earlier slices will reach a higher peak intensity before being defocused. These time-dependent focusing and defocusing processes lead to the temporal reshaping of the pulse. As seen in Fig. 3.1b, there is a sharp leading edge with a smoother and very broad back component. This latter spatially broad low energy distribution of the field constitutes partially the background reservoir. It will exist persistently in all the following results so long as there is self-focusing. The front part of the pulse is now shorter than the original pulse length because the back part is diffracted by the plasma (Fig. 3.1b). This is how self-pulse compression comes by. With further propagation a second pulse appears at the back of the leading pulse which becomes even shorter, as seen in Fig. 3.1c and d. This is what has been called pulse splitting. We should call this refocusing. That is to say, the back part of the pulse, after being diffracted by the plasma left behind by the front part of the pulse, contains sufficient energy and power so that self-focusing restarts after some distance of propagation. The peak at the back part of the pulse keeps increasing as the pulse keeps propagating as shown in Fig. 3.1d.

In Fig. 3.2a, the normalized filament energy (defined as the energy contained in the filament core within a diameter of 150-μm to the total pulse energy) is plotted as a function of the normalized propagation distance, $z/z_0$ (Aközbek et al., 2001). Initially, due to self-focusing, more energy of the pulse is channeled into the core region until there is enough plasma generated to stop the self-focusing process, and the beam starts to defocus. However, the defocusing is stopped and the pulse refocuses again, which can be seen as the second peak in the filament energy. This process can repeat itself many times which is apparent from Fig. 3.2a as a weak third peak in the filament energy. This multiple refocusing phenomenon was observed experimentally (Fig. 2.5 and see also Chin et al., 2005). The simulation and the experiment are in good qualitative agreement with each other.

Figure 3.2b shows the generated linear electron density along the propagation direction. It clearly agrees with the refocusing discussed in the filament energy description in Fig. 3.2a. Whenever the pulse refocuses, more electrons are being generated which are seen as the peaks in Fig. 3.2b and their location agrees well with the peaks in the filament energy depicted in Fig. 3.2a. Alternatively, one can examine each temporal slice of the temporal intensity profile as a function of propagation distance. Figure 3.2c shows the on-axis intensity $I(r = 0, \tau = 0)$ of the temporal slice which has the highest initial peak power. It will come to a focus
Fig. 3.2  (a) Refocusing as seen from the plot of the normalized energy inside the filament core versus the normalized propagation distance $z/z_0$. The normalized energy is defined as the ratio of the energy contained inside a diameter of 150 $\mu$m centered around the filament axis to the total pulse energy. (b) Re-focusing as seen from the electron density plot. (c) Re-focusing as seen from the intensity plot (courtesy of Dr. Neset Akozbek)
first and the peak intensity increases until plasma defocusing stops the self-focusing process and it starts to defocus; but it only defocuses until self-focusing takes over again. This process can take place many times resulting in multiple self-focusing collapses. The refocusing of the pulse channels energy back into the core of the beam and thus represents the process in which energy is exchanged between the core and the outer part (background reservoir) of the beam. This is one of the important physical mechanisms of the long-range propagation and filament formation in air.

At the end of the propagation when $\frac{z}{z_0} = 1$, the front part of the pulse becomes less important than the back part of the pulse. The latter becomes very steep at the back side with a negative slope; i.e., self-steepening. This is seen in Fig. 3.1e. This steepness naturally gives rise to a very broad spectrum toward the blue side of the main frequency of the initial pulse. That is, it will result in a huge spectral broadening of the pulse; i.e., a supercontinuum or white light laser (see Chapter 2.). This is when the filament becomes mature (Section 2.5).

It is worth emphasizing that the popularly called supercontinuum is not a separate pulse generated by filamentation but is simply the same pulse which self-transforms into a white light laser pulse with a very steep back part.

Many other methods have been used to derive the nonlinear Schrödinger equation in one form or another (see the reviews in Chin et al., 2005; Couairon and Mysyrowicz, 2007; Bergé et al., 2007; Kasparian and Wolf, 2008 and the references therein). All of them give similar results as explained above.

### 3.4 Filamentation in Condensed Matter

In principle, the physics of filamentation in condensed matter is similar to that in air or other gases. One major difference is the generation of free electrons at the self-focus. Instead of pure tunnel ionization of gas molecules, in condensed matter, the generation of free electrons starts from the excitation of electrons from the valence to the conduction bands (Brodeur and Chin, 1998) followed by inverse Bremsstrahlung and a few cycles of collisional ionization because the density of condensed matter is high (see Chapter 1). Equation (3.7) will have to be modified. For more details, see Chin et al. (2005) and Kandidov et al. (2003a).

### 3.5 x-Wave and Conical Emission

Recently, there have been many papers treating filamentation as x-waves generation (see for example, Di Trapani et al., 2003a; Kolesik et al., 2004). Strictly speaking, x-wave is a class of mathematical solution of the wave equation under certain constraints and is related to the solution of solitary/localized wave packets (Claudio Conti, 2005). It could be linear or nonlinear. In the nonlinear optical case, it is related to the so-called light bullet first coined by Silberberg (1990). The reason

---

1This section was written with the kind collaboration of Dr. Olga Kosareva of Moscow State University.
why it is called x-wave is probably because when the solution is expressed as a kind of contours of “intensity” distribution in a 2D plot with two parameters (normally spatial-temporal or spatial-spectral), the resulting picture (2D distribution) looks like an x-letter in the ideal case. Such x-shapes indicate the conical structures of the wave-object with cylindrical symmetry in the propagation direction.

In the case of filamentation, Kosareva and Kandidov of the Moscow State University (see Kosareva et al., 1997a, b) are among the firsts, if not the first, who have expressed the solutions of filamentation in the form of x-shaped structures (see also Golubtsov et al., 2001; Kandidov et al., 2003a, 2004). However they did not use the word x-wave because the focus of their work was not on localized wave packet. Rather, they used the 3D representation of the pulse’s spatial-temporal as well as spatial-spectral intensity distribution to illustrate the physics of filamentation, white light laser (supercontinuum) and conical emission.

Let us look at their computer solution of the wave equation. Figure (3.3, top) shows, in a 3D semi-logarithmic plot, the spatial-spectral intensity distributions

![Spatial and spectral intensity distribution](image)

**Fig. 3.3** Spatial and spectral intensity distribution of a 230 fs(FWHM)/800 nm pulse of radius $a = 167 \mu m$ after propagating a distance of $z = 52 m$ in the atmosphere. Top: 3D representation; bottom: 2D representation. For detail of the propagation physics, see Golubtsov et al. (2001, Fig. 7). This is an original unpublished plot generously given to the author by Olga Kosareva, Moscow State University
3.5 x-Wave and Conical Emission

Fig. 3.4 Spatial and temporal intensity distribution of the same pulse at propagation distance of $z = 52$ m in air. Top: 3D representation (in linear scale); bottom: 2D representation (in semi-log scale). Note the x-nature of the plot. For detail of the propagation physics, see Golubtsov et al. (2001, Fig. 7) (courtesy of Olga Kosareva, Moscow State University)

The normalized intensity $S(\theta, \lambda)$ of a fs laser pulse after propagating in the atmosphere for 52 m in the z-direction ($z = 52$ m). The normalized intensity $S(\theta, \lambda)$ in log-scale is expressed as function of the divergence angle (transverse spatial coordinate with cylindrical symmetry around angle zero) and the wavelength (spectral coordinate). This (as well as Fig. 3.4 to be discussed later) is an extension of the 3D-plots in Golubtsov et al. (2001, Fig. 7) and was done by Dr. Olga Kosareva of the Moscow State University. There is an extension of the intensity scale down to $10^{-11}$ showing a long tail at the short wavelength side in Fig. 3.3. In the original paper (Golubtsov et al., 2001), this scale was only down to $10^{-10}$.

This 3D x-shape (or rather v-shape) structure characterizes conical emission. Conical emission of different colors (see Chapter 2.) is obtained in Fig. 3.3 (top).
Conical emission in the 3D plot (top) can be visualized by recognizing that the 3D object has a radial symmetry. Making a revolution of the plot around the central axis \((\theta = 0)\) would give rise to some ring structures in the 3D plot of Fig. 3.3 (top). Ring structure corresponding to the major v-shape tail shows up at the shorter wavelength side of the fundamental wavelength (800 nm). This corresponds to conical emission toward the blue side of the fundamental as discussed in Chapter 2. These rings at the blue side of the fundamental (see Fig. 2.1) come predominantly from the interaction and diffraction by the plasma at the front part of the pulse (Kandidov et al., 2003a, Fig. 4). Toward the red side of the fundamental, there are weaker v-shape local maxima indicating propagation of the pulse in the Kerr medium with material dispersion giving rise to rings (conical emission) (Luther et al., 1994). There should also be rings symmetrically located at the blue side of the fundamental due to the same effect if no plasma were generated. However, because of plasma generation, rings on the blue side are due to the combined effect of the dispersion and the plasma; therefore we do not have complete symmetry between the blue and the red sides of the spectrum.

Now, when the 3D plot is projected into a 2D contour plot, a complex x-shape structure, stronger at the blue side and weaker at the red side of the fundamental is revealed (Fig. 3.3, bottom). This x-shape object in filamentation is called x-wave by some (see for example, Di Trapani et al., 2003a) in an attempt to relate it to the universal dream of non-spreading wave packet. One should note that the x-tails are very weak because the plots are in log-scale. We also note that the x-wave in Fig. (3.3, bottom) looks almost the same as the x-wave obtained in water by Kolesik et al. (2004, Fig. 4). The difference between Fig. (3.3, bottom) and Fig. 4 from Kolesik et al. (2004) is mainly due to lower material dispersion in air. In particular, in air, we may not see the secondary system of “horns” at the red side of the fundamental, which, although weak, are clearly pronounced in Kolesik et al. (2004; see below).

The two secondary, weaker “horns” at the blue side of the fundamental appearing in the 2D representation or the secondary system of rings in 3D representation are associated with the trailing sub-pulse at this position of propagation of \(z = 52\) m. Figure 3.4 (top) shows the intensity distribution in space and time in a 3D plot. The peak intensity of the trailing sub-pulse is higher than that of the leading sub-pulse at this propagation distance. Thus, the trailing sub-pulse creates its own system of (weak) conical rings. In principle, the trailing sub-pulse should create rings toward both the blue and red sides of the fundamental. However, the trailing sub-pulse “front” is not so steep as the real front of the leading sub-pulse and the dispersion in air is low. Therefore, secondary “horns” at the red side of the fundamental in air are absent. If the dispersion in air were stronger, one would have exactly the same picture as Kolesik et al. (2004) who had chosen water for their simulations and x-wave generation. Note that each sub-pulse whose intensity growth is stopped by either material dispersion or the plasma forms its own x-pulse in the spatial-spectral domain. However, the ring systems from different sub-pulses may not coincide exactly. This is because the sub-pulses are produced at different propagation distances and have slightly different divergence.
The projection of the 3D representation onto a 2D representation shows again a quasi x-shape object; i.e., again an x-wave, now in the spatial-temporal domain (Fig. 3.4, bottom). In fact, even before pulse splitting, the 2D representation of the filamenting pulse already shows the x-wave structure in the simulation by Kosareva et al. (1997a, Fig. 2).

Along the axial direction toward the shorter wavelength side in the 3D plot of Fig. (3.3, top), there is a long on-axis tail mainly due to self-steepening (Kandidov et al., 2003a, Fig. 4; Aközbek et al., 2001). Thus, the combined on-axis spectrum would be very broad and would appear as a “white” spot on the transverse pattern (Fig. 2.1) after filamentation and is the basis for what we call by “white light laser” or supercontinuum. At the fundamental wavelength in the 3D plot (Fig. 3.3, top), there is an extended ridge with many local maxima extending toward large diverging angles. These local maxima would represent the weak ring structure of the fundamental after propagation (see Fig. 2.19).

In addition, if the equi-intensity contours in the 2D plots are changed into color coding, thanks to modern computational advancement, the 2D plots become beautifully attractive turning into a piece of art (Di Trapani et al., 2003b). But they are still the same spreading pulse during filamentation. At this stage, no more new physics could be learned from the x-wave representation from the point of view of filamentation, since we have already discussed all the features of such propagations including conical emission.

Again, interested readers are referred to Couairon et al. (2006) for a review of the theory of x-waves in filamentation and Xu et al. (2008) for an experimental manifestation of the x-wave and references therein. More detailed physics has already been explained in the “natural” way (i.e., without using x-wave presentation) by Kosareva et al. (1997a, b), Kandidov et al. (2004), Golubtsov et al. (2001), Kandidov et al. (2003a), Chin et al. (2005), Couairon and Mysyrowicz (2007), and Bergé et al. (2007).